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SELECTED RESEARCH INTO PREDICTIVE MECHANISMS FOR ANTISUBMARINE WARFARE

by

Peter K. Luster
The John D. Kettelle Corporation

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THE JOHN D. KETTELLE CORPORATION
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13. ABSTRACT

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Search Theory						

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ABSTRACT

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I. INTRODUCTION

A. PREFACE

This paper develops four topics related to prediction in ASW. They are collected together here because the topics are commonly related to game theory and Bayesian techniques.

Parts of this paper use the terminology and results presented in a previous paper, Contingency Trees, Contingency Plans, and Utility Functions.

Chapters II and III of this paper describe some results of the search for ways to evaluate Bayes' formula which are economical of compute time. Chapter IV explores some of the relationships and mutual benefits of game theory and Bayesian techniques. The appropriateness and problems of applying general-sum game analysis is discussed. The final chapter, Chapter V, proposes a precise measure of the amount of information in data about the location of objects being searched for. This measure has potential as an objective measure of the efficiency of predictive mechanisms.

B. BAYES' THEOREM

The terminology and nomenclature of Bayes' theorem (or formula) are set down below for reference from the later text. Bayes' theorem in the form used here states:

$$P[H_i|D] = \frac{P[D|H_i]\ P[H_i]}{\sum_{j=1}^{n} P[D|H_j]\ P[H_j]} \qquad (1)$$

where $\{H_i \mid i = 1,\ldots,n\}$	is a set of n mutually disjoint hypotheses which exhaust the possibility space.
D	is an observed event or datum.
$P[H_i]$	is the prior or <u>a priori</u> probability of the ith hypothesis, i.e., before the observation of D.
$P[H_i\|D]$	is the posterior or <u>a posteriori</u> probability of the ith hypothesis, i.e., after the observation of D.
$P[D\|H_i]$	is the probability of event D, given the hypothesis H_i.

II. ECONOMIZING APPROXIMATIONS FOR BAYESIAN PROCESSORS

A. GENERAL COMMENTS

A serious drawback to the application of Bayesian techniques to certain classes of situations is the large amount of computation needed to evaluate Bayes' formula if the number of hypotheses is large. This chapter presents an approximation technique for the evaluation of Bayes' formula which reduces the computational load for these situations and which permits any desired closeness of approximation for a single application of Bayes' formula. Behavior of the error over more than one application of Bayes' formula is yet to be studied.

B. THE SEARCH FOR n MOVING OBJECTS

A particular situation of interest is using a Bayesian processor in a search for *n* moving objects in a two-dimensional surface. Under movement assumptions like those in [Luster, et al, 1968], the number of hypotheses H that a Bayesian processor must consider increases exponentially with *n* and likewise increases exponentially with the distance[1] through which a given object is allowed to move. Therefore if *n* and/or the distance (more precisely, the number of track segments) is at all large, the number of hypotheses to be considered in a straightforward application of Bayes' Theorem rapidly becomes huge, and the calculations for evaluating their posterior probabilities are too voluminous to be feasible.

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[1]More precisely, with the number of track segments representing the objects' movements; the larger the number of segments, the greater the detail in which the movement is studied.

C. THE APPROXIMATION

A solution to this problem is had by restricting the *a priori* probabilities P[H] and the conditional probabilities P[D|H] to a relatively small number of values. Upper and lower bounds on these quantities are chosen from among the selected values, and the corresponding upper and lower bounds on the posterior probabilities P[H|D] are calculated. The saving in computation time comes about because all computations fall into one of several distinct categories. The categories are defined by the selected values included in the calculations; because these values are restricted at the outset, only a few such categories result. Thus only one calculation need be carried out for each category; subsequent cases which fall into that category will already have had their result calculated, which can be simply added into the appropriate tally.

This procedure may be used iteratively so as to "zero in" on the correct values of the hypotheses that preliminary analysis shows to be of greatest interest. Note that one is dealing with upper and lower bounds throughout, so that the true value of P[H|D] is known to lie between the two bounds, and so that the difference between the two bounds gives an upper bound on the error.[2] If this error is too large, the process can be repeated, choosing closer bounds at the outset. Note that the selected values need not be selected at equal intervals in the zero to one range.

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[2] If the midpoint between the upper and lower bounds is used to estimate P[D|H], the maximum error is at most half this difference in absolute value.

III. TIME TO COMPUTE BAYES' FORMULA USING HARDWARE

A. GENERAL COMMENTS

Special computing hardware has been suggested as a way to evaluate Bayes' formula, obviating the problems of extensive calculations required in certain Bayesian processors as discussed in the previous chapter. Research has been conducted in an attempt to derive a lower bound on the time required by a calculating device to perform such an evaluation of Bayes' formula. This research has been successful and its results summarized in this chapter.

These results apply to a single application of Bayes' theorem. Providing certain conditions of independence are met, iterative applications are permissible and are additive in terms of time to compute. Research has not been carried out for time to compute in the case of sequential application of Bayes' theorem for the non-independent case, but it appears to present no insurmountable problems.

B. SUMMARY OF THE RESULTS

Suppose that the possible values of $P[D|H_i]$ can be adequately approximated by choosing from among the finite set $\{p_1, p_2, \ldots\}$. Then in any given finite number of iterations of Bayes' theorem, the possible values of $P[H_i|D]$ will likewise be finite, though perhaps large. They may, however, themselves be approximated by some, perhaps smaller, set of values, among which the observed data D cause changes to take place. As in [Luster, 1970] the possible changes may be represented as due to the action of a group operation. The time required to compute Bayes' theorem may therefore be estimated by an estimate of the time required to carry out the corresponding group operation.

Suppose that the possible values of $P[H_i|D]$ are approximated by some set of N values. Then there will be a maximum of N^{m-1} combinations of values that the $\{P[H_i|D]\}_{i=1}^{m}$ can assume at any one time; the exponent of N is "m-1" instead of "m" because the posterior probabilities must sum to 1. Then the group G mentioned in the last paragraph may be taken to have N^{m-1} elements.

Applying [Winograd, 1965, Section III, Theorem 1], the time required to compute an operation in a group G is at least $\lceil \log_r 2\lceil \log_d \alpha(G)\rceil \rceil$ time units. Here "$\lceil x\rceil$" denotes the smallest integer not less than x. The function $\alpha(G)$ is the order of the largest subgroup H of G having the property: either $H \equiv$ {the identity element} or else there exists a nonidentity element $a \in H$ such that every proper subgroup of H contains a. The d and r are parameters representing hardware characteristics and are defined in [Winograd, 1965].

C. AN EXAMPLE

Suppose that the possible values of $P[H_i|D]$ are approximated by the ten values $\{\frac{1}{20}, \frac{1}{10}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{9}{10}, \frac{19}{20}\}$. It is assumed, as in [Luster, 1970], that suitable "overflow" and "underflow" indicators are available to sense the penetration of an "upper threshold" above $\frac{19}{20}$ or of a "lower threshold" below $\frac{1}{20}$, at no extra expense in computing time. Suppose that 10,000 hypotheses H_i are of interest (corresponding, say, to a 100 x 100 array of cells in which it is desired to locate given objects). The order of the group G corresponding to this situation is 10^{9999}. Assuming G has been defined so as to be commutative, $\alpha(G)$ is at most 5^{9999}. For d = r = 2, the lower bound is:

$$\lceil \log_r 2 \lceil \log_d \alpha(G) \rceil \rceil \leq 1 + \lceil \log_2 \lceil \log_2 5^{9999} \rceil \rceil = \text{16 time units.} \tag{2}$$

This time value can be compared to the value obtained for the case where the posterior probabilities are allowed two values and the number of hypotheses are two. In this case, the time to compute Bayes' formula $\geq$ two time units. Using currently available integrated circuits, the time to compute this latter case will be on the order of 100 nanoseconds. This implies a lower bound of less than a microsecond for the former case involving 10,000 hypotheses.

While these are lower bounds, [Winograd, 1965] indicates that actual circuit times approach these bounds as the complexity of the logic components used in the circuits increases. For the bounds computed above, the simplest logic components were used; it was assumed each component has two input lines and one output line, and each line can be in at most two possible states.

Time required to encode the observed data and to retrieve and store the probabilities has not been included. However, the times to perform these two operations are additive to the time to compute developed above.

These results suggest the possibility of tremendous saving in computation time for a Bayesian processor by use of hardware components. There is, though, need to investigate the behavior of the relationship of amount of hardware to the numbers of hypotheses, distinguishable observed data, selected numeric representations of probabilities, and the time to encode the observed data and to retrieve and store the probabilities. Once this relationship is understood, it should be possible to determine trade-offs between time to compute and hardware costs.

IV. RELATIONSHIPS AMONG GAME THEORY, BAYES' THEOREM, CONTINGENCY PLANS, AND STRATEGY

A. GENERAL REMARKS

In the past there has been discussion of the relative merits, redundancy, and uniqueness of Bayesian processors and game theory. The following paragraphs present, first, a discussion of the individual merits of game theory and the meaning of strategy in terms of general sum games. It is concluded that general sum games will be an important tool of operations research in the future.

The language of contingency trees, contingency plans, and their utilities is used. A previous paper in this series develops this language.

Finally, by way of an example, a symbiotic relationship between game theory and Bayesian techniques is presented. It is concluded that such a combined approach has power to handle a number of military decision-making situations.

The final example suggests a powerful way to use game theory to support Bayesian processing. One of the difficulties in applying Bayesian techniques has been the frequent lack of a rationale for estimating a priori probabilities. Game theory exhibits the potential for supplying that rationale.

B. GAME THEORY AND STRATEGIES

The intersection of two complete contingency plans determines a path in a contingency tree. In the language of game theory, complete contingency plans correspond to (pure) strategies, and paths correspond to states of the world determined by the choice of a (pure) strategy by each player in a two-person game. It would be easy to modify the definition of a path so that a path would be determined by the intersection of any given number n of complete contingency plans, taken one each from each of n respective classes of complete contingency plans. Then the game-theory correspondence would be to an n-person game. But it is unnecessary for present purposes to make this generalization. Thus, in game-theory language it is a "player" who may choose a (pure) strategy (= complete contingency plan) and there are as many players as there are sets of (pure) strategies (= classes of complete contingency plans) to be chosen from, one (pure) strategy from each set of strategies. The utilities of the various states of the world may, and generally do, differ for each player. A two-person game is zero-sum if there is some constant c such that for any given state of the world, the utility of that state of the world for the first player plus the utility of that state of the world for the second player equals c.

A (not necessarily pure) strategy results in game theory when a player elects to choose from among his available pure strategies by using some chance device with known probability characteristics. That is, he elects to choose (pure) Strategy 1 with probability p_1, (pure) Strategy 2 with probability p_2, etc., where $p_1 + p_2 + \ldots = 1$. This election is itself a strategy; it is equivalent to a pure strategy if one of the p_i's is 1, and is otherwise termed a "mixed strategy."

A maximin strategy for a player in a two-person, zero-sum game is one which maximizes his minimum expected utility, where the minimum[1] is taken over all possible strategies of the other player. As remarked in Section D below, if a player in a two-person zero-sum game supposes that his opponent will play his opponent's maximin strategy and if the player chooses his own strategy so as to maximize his own expected utility on the basis of this supposition, then (if the supposition is correct) the player will obtain an expected utility equal to the expected utility for his own maximin strategy. If, however, the player supposes that his opponent will play some other specified strategy, and if the player chooses his own strategy so as to maximize his own expected utility on the basis of this supposition, then (if the supposition is correct) the player may obtain an expected utility greater than the expected utility for his own maximin strategy. If the supposition is wrong, he may do worse. Section D suggests how the player can hedge: he can ensure against unacceptable loss if his supposition is wrong by accepting a lesser gain if his supposition is right.

The preceding paragraph shows a relationship between estimated a priori probabilities of the opponent's choice of pure strategies and the opponent's maximin strategy in a two-person zero-sum game. The player has some reason to think the opponent will play the opponent's maximin strategy. The player's own maximin strategy will maximize the player's expected utility given that the opponent plays the opponent's maximin strategy. Other strategies may likewise maximize the player's expected utility given that the

[1]When the number of strategies available to each player is finite, this minimum exists, as is easily shown.

opponent plays the opponent's maximin strategy; but, unlike the player's maximin strategy, these other strategies may result in loss to the player if the opponent departs from the opponent's maximin strategy. Thus, although the player has some reason to think that his opponent will choose pure strategies as if with a priori probabilities determined by the maximin strategy, maximizing the player's expected utility under this supposition is not in itself sufficient to ensure optimal (in the sense of maximin) play on the player's part.

The situation becomes much more complex in the case of possibly non-zero-sum, or general-sum, games. Here the player can simply ignore his opponent's utilities and choose his own maximin strategy. He may, however, be able to do much better, depending on the arrangement and values of the utilities and, perhaps, on such things as the availability of channels of communication, and his opponent's willingness to cooperate.

A part of the problem of general-sum games is to define what is meant by an optimal strategy in a given set of circumstances. Practical difficulties then typically arise in trying to decide whether or not a given real-life situation fits the circumstances for which a given strategy is, according to the theory developed, optimal. Hence general-sum games provide examples of situations where it is difficult to say with what a priori probabilities a player should (normatively) adopt each of his pure strategies, and where it is likewise difficult to say with what a priori probabilities a player will (empirically) adopt each of his pure strategies.

These difficulties notwithstanding, and perhaps in part because the difficulties reflect problems that are an essential part of the real world that general-sum games attempt to model, the theory of general-sum games promises to play an increasingly prominent role in operations research in the decades ahead. As has just been noted, problems in assigning *a priori* probabilities can reflect the problems of defining an optimal strategy for general-sum games. Progress in the latter field, both normative (how *should* players play) and empirical (how *do* players play), should help in assigning values or ranges of values of *a priori* probabilities in game-like situations.

Warfare (in the broader sense which includes the prevention of wars) is particularly rich in situations where cooperation can be mutually beneficial and lack of cooperation mutually disastrous.

The First World War is a haunting example. Through the years of the war, the participating nations lost millions of their young men. Following the war, short-sighted political forces constructed a peace that destroyed the stability of German government and which directed Europe into economic collapse. Communist designs threatened or conquered nations weakened politically and economically by the war. Two personalities, Hitler and Mussolini, nourished Fascist politics by founding their popular appeal on reaction to Communist threat and economic disease.

Whether immediate effects or historical consequences are assessed, there were no victors in this war. The "players" lacked the ability to recognize or the mechanism to effect a strategy which would have benefited all.

The post-war politics repeatedly failed to turn the course of history away from the second war; in fact they appear in retrospect to have accelerated development of the causative conditions. While they did recognize that some interactions were non-zero sum games (for example, the game whose selection of strategies is represented by the Munich Agreement), they failed to recognize a broader non-zero sum game. In The Gathering Storm Churchill identifies this kind of suboptimization:

> We shall see how the counsels of prudence and restraint may become the prime agents of mortal danger; how the middle course adopted from desires for safety and quiet life may be found to lead direct to the bull's-eye of disaster.

C. USE OF PARTIAL CONTINGENCY PLANS - AN EXAMPLE

Previous work[1] provides an example of a partial contingency plan adequate to fulfill a prescribed mission--namely, keeping the evader within a specified region of pursuer relative space. For this example, the members of the contingency tree $\mathscr{A}$ may be taken to be all possible events of the form: the track (including both positions and time at each position) of the evader in pursuer relative space contains the initial track segment...[specification of initial track segment, including both positions and time at each position].... Here the initial track position and starting time is assumed to be given and fixed. $\mathscr{A}$ is rooted at the event A_0: the track of the evader in pursuer relative space contains the [degenerate] initial track segment...[consisting of the evader's initial track position and starting time].... The partial contingency plan specifies what the pursuer is to do whenever the evader reaches certain critical points in pursuer relative space; it is not a complete contingency plan (i.e., not a "strategy" in the usual game-theory sense) because it does not say what the pursuer is to do otherwise.

Let the utility of any path $\mathscr{P}$ in $\mathscr{A}$ be 1 if the pursuer never leaves the prescribed area, given $\mathscr{P}$, and 0 otherwise. Then following the partial contingency plan assures that the utility will be 1, regardless of the evader's strategy.

The notion of a partial contingency plan seems closer to the notion of "strategy" (as opposed to "tactic") in the non-game-theory use of the term. "Strategy" usually connotes

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[1] See [Dodson, Luster, and Randolph, 1970, Section 4.6].

a broad, overall plan, with details still having to be filled in later. Obviously such general approaches are invaluable in practical applications. Partial contingency plans may therefore find a useful practical application at the overall planning level for which complete contingency plans, owing to their excessive concern for detail, may be unsuited.

D. GAME THEORY AND BAYESIAN TECHNIQUES - AN EXAMPLE

1. Introduction

Combined use of game theory and Bayesian techniques enjoys a synergistic relationship when applied to certain kinds of problems. This section presents an example of their combined application to the problem of decision-making under uncertainty. The problem is cast in a military situation, that of a commander of battlefield units. While the situation is simple, it is qualitatively similar to many tactical situations faced by decision makers in all three services.

2. The Setting

A commander finds himself in the following tight situation. It is late evening. His men are tired and greatly outnumbered. The country is mountainous, jungle covered, and largely in control of the enemy. Reinforcements are due in by helicopter the first thing in the morning. Meanwhile, his men have only just had time to dig in for an expected attack that night.

Because of the terrain, attack can come only from two directions: straight ahead, or on the exposed right flank. The enemy is known to have insufficient firepower to support strong attacks from both directions simultaneously. But the commander also has insufficient firepower to defend against strong attacks from both directions; and the terrain offers no positions for mounting the same guns so as to defend against attacks from either direction.

3. Game Theory Analysis

The probability of being overrun that night depends on where the enemy mounts a strong attack and where the defending commander concentrates his firepower. The commander's estimate of this relationship is shown in Table 1.

Table 1. PROBABILITY OF BEING OVERRUN, GIVEN EACH COMBINATION OF STRATEGIES

Commander Disposes Main Fire Power So As to Defend	Enemy's Main Attack is Against	
	Front	Right Flank
Front	.10	.70
Right Flank	.50	.30

These probabilities are based solely on the commander's judgment and experience. The effect of changes in these probabilities could be determined, if desired, by applying the methodology to be presented to the changed probabilities.

Applying elementary game theory to the estimates in Table 1 yields optimal strategies of

$$p_1 = .25 \qquad q_1 = .50$$

$$p_2 = .75 \qquad q_2 = .50, \tag{3}$$

where

p_1 is the probability with which the commander disposes his main firepower so as to defend the front

p_2 is the probability with which the commander disposes his main firepower so as to defend the right flank

q_1 is the probability with which the enemy mounts his main attack against the front

q_2 is the probability with which the enemy mounts his main attack against the right flank.

To see what is meant here by "optimal" let

$$\begin{aligned} \delta &= p_1 - .25 = .75 - p_2 \\ \varepsilon &= q_1 - .50 = .50 - q_2 \end{aligned} \tag{4}$$

where

$$-.25 \leq \delta \leq .75$$

$$-.50 \leq \varepsilon \leq .50.$$

It follows from Table 1 that the probability P of being overrun may be expressed as a function of δ and ε as

$$\begin{aligned} P = \; & .10(.25 + \delta)(.50 + \varepsilon) + .70(.25 + \delta)(.50 - \varepsilon) \\ & + .50(.75 - \delta)(.50 + \varepsilon) + .30(.75 - \delta)(.50 - \varepsilon) \\ = \; & .40 - .80\delta\varepsilon. \end{aligned} \tag{5}$$

By choosing $\delta = 0$, the commander ensures that the probability of being overrun will be neither greater than nor less than .40. Similarly for the enemy choosing $\varepsilon = 0$. Other choices of δ and ε lead to a change in the probability of being overrun: the probability becomes less if δ and ε are both positive or both negative, and greater otherwise. Choosing $\delta = 0$ is the only way that the commander can be certain that the probability of being overrun is not greater than .40. This is the sense in which $\delta = 0$ in his optimal strategy.

4. Use of Bayesian Analysis

Table 1 of the preceding section summarized the commander's view of the situation, based on his own judgment and experience. There is no guarantee that the enemy's summary of the situation would look like Table 1 or, if it did, that he would apply game theory in deciding on a course of action. These considerations suggest that if the enemy's actual strategy amounted to choosing $\varepsilon = 0$, it would be something of a coincidence--even crediting the enemy with a knowledge of game theory. Hence if the commander can be satisfied that the enemy is more, or less, than fifty per cent likely to attack the front, he may choose some $\delta \neq 0$ in an attempt to decrease the probability of being overrun from his estimated .40.

Suppose that the commander receives intelligence reports which make it appear very likely that the enemy will mount its main attack in the front and that the enemy's firepower will be concentrated so as to support a frontal attack. The commander estimates that the probability of receiving these reports is .80 if the enemy really is going to mount a frontal attack. If the enemy is going to attack the right flank he estimates the probability of receiving these same reports is .10.

The intelligence reports clearly are relevant to the commander's decision as to where to locate his own firepower. But just how should he use the intelligence reports?

Let q_1 and q_2 be the commander's estimates, before receiving the intelligence reports, of the probability that the enemy will mount its main attack on the front or on the right flank, respectively. Given q_1 and q_2 Bayes' theorem provides a way of obtaining revised probabilities $q_{1'}$ and $q_{2'}$ of these events by taking the intelligence reports into consideration. Applied to the present situation Bayes' theorem states:

$$q_{i'} = \frac{P(R|i)q_i}{P(R|1)q_1 + P(R|2)q_2} \qquad \text{for } i = 1,2 \tag{6}$$

where $P(R|1)$ is the probability of the commander receiving the intelligence reports given that the enemy is going to mount a main frontal attack and $P(R|2)$ is the probability of receiving the intelligence reports given that the enemy is going to mount a flank attack.

The commander might take the values of q_1 and q_2 calculated in the preceding section as his estimate of their actual value. Then, applying Bayes' theorem:

$$\begin{aligned} q_{1'} &= \frac{(.80)(.50)}{(.80)(.50) + (.10)(.50)} = .89 \\ q_{2'} &= \frac{(.10)(.50)}{(.80)(.50) + (.10)(.50)} = .11. \end{aligned} \tag{7}$$

In the notation of the preceding section, the values of $q_{1'}$ and $q_{2'}$ correspond to a value of

$$\varepsilon = .39 > 0. \tag{8}$$

Accordingly any value of δ satisfying

$$0 < \delta \leq .75 \tag{9}$$

will decrease the probability P of the commander's troops being overrun, with the decrease becoming more pronounced as δ is increased. The lowest value of P thus corresponds to $\delta = .75$ (i.e. $p_1 = 1$ and $p_2 = 0$); it is

$$P = .40 - .80\delta\varepsilon = .40 - .80(.39)(.75) = .17. \tag{10}$$

The validity of this sizable reduction in P depends on both the initial estimates of q_1 and q_2 and the analysis involving the estimated accuracy of the intelligence reports. Had the enemy somehow been able to give the appearance of attacking from the front, whereas in reality he was planning to attack on the right flank, the value of P would change from .40 to .70 in the event that $p_1 = 1$. This would be the cost of being deceived by the enemy. This possibility has already been taken care of in the commander's estimate that $P(R|2) = .10$. Nevertheless the commander may want to hedge further against the possibility of being caught with his main firepower defending the front when the main attack is on the right flank. At the same time he may want to take some advantage of the intelligence reports so as to reduce P.

One way of hedging further would be to set an upper bound (say .50) on the "worst case" probability of being overrun. The Bayes' theorem estimate of $\varepsilon = .39$ could then be used to choose δ so as to minimize P subject to the constraint that P is not to exceed .50 for this δ and for <u>any</u> ε between -.50 and .50.

Some positive value of δ will fulfill these conditions. Given a positive δ, the worst (from the commander's point of view) possible value of ε is the negative one largest in magnitude--viz., $\varepsilon = -.50$, corresponding to $q_1 = 0$ and $q_1 = 1$. For this worst case

$$P = .40 - .80\delta(-.50) = .50,$$

so that $\delta = .25$, corresponding to $p_1 = .50 = p_2$. Using this strategy the commander will have probability

$$P = .40 - .80\delta\varepsilon = .40 - (.80)(.25)(.39) = .33 \qquad (11)$$

of being overrun if the analysis regarding the intelligence reports is correct. Moreover the probability of being overrun will not exceed .50 even if the analysis regarding the intelligence reports is wrong and even if the enemy adopts an optimal strategy.

5. Final Comments

The foregoing discussion gives an example of the synergistic use of game theory and of Bayes' theorem in a problem which, though simplified, contains a number of realistic elements. The discussion shows how a battlefield commander might make a decision under uncertainty and how he might use additional intelligence which becomes available to him about the enemy's actions.

The example also shows the essential difference between the game theory and Bayesian techniques. Game theory is used to decide what to do based on the information available; Bayesian techniques are used to process the available data, producing thereby the information input to the decision-making procedure.

Finally, the example shows game theory as a useful adjunct to Bayesian processors as a technique for estimating *a priori* probabilities.

V. A MEASURE OF INFORMATION FOR SEARCH THEORY

A. GENERAL COMMENTS

Theoretical and practical research of ASW predictive mechanisms and more general search problems need a measure of the value of data about the objects searched for. This value, termed here the "amount of information" contained in the data, is useful in measuring (1) the relative performance of several competitive predictive mechanisms, (2) the efficiency of procedures which produce *a priori* probabilities, and (3) the value of the contribution or potential contribution of sensors and tactics to a search effort, as examples.

B. THE MEASURE

The measure of information is essentially the expected time to find the first of the objects being sought. The following text assumes this definition, although related definitions, e.g., expected time to find all objects sought, can be considered analogously. This measure is of interest, not only because of its obviously close relation to real-world operations, but also because it may be simply and directly related to objective functions or constraints on the basis of which decisions must be made.

By way of specific context assume that n objects are distributed among a given set of cells, and that the probability of an object being in each cell is known. Then the measure of information of this data (i.e., the probabilities) is the expected time to find the first of these objects given the probabilities are known.

Clearly the type of search to be conducted plays an important role here. To be widely applicable, the methods of search available must be somewhat idealized; to remain useful, they must be appropriately idealized, and the idealization must remain within reasonable bounds. In some circumstances the idealization can be carried out in such a way that the expected search time is closely related to Shannon's familiar measure H of information or uncertainty. In the applications of interest in the present paper a different idealization seems appropriate.

Suppose that there are just m given cells and that the *a priori* probability of there being at least one object in cell j is p_j, for $j \in \{1,\ldots,m\}$. Given that there is at least one object in some specified cell, suppose that the probability of having failed to find any object in that cell after searching for t time units is q^{rt}, where r is the proportion of the available search force used for searching the specified cell and q is a constant ($0<q\leq 1$) characterizing the capability of the search force and the difficulty of the search. Relabeling if necessary, let $p_1 \geq p_2 \geq \cdots \geq p_m$. For standardization, let $q = e^{-1}$. Then it can be shown that, using an optimal search strategy, the expected search time required to find the first object is

$$\sum_{j=1}^{m} [2j - 1 + \ln \frac{\prod_{k=1}^{j-1} p_k}{p_j^{\,j-1}}] p_j. \qquad (12)$$

Because of the standardization, the expected search time is independent of q. To obtain from (12) the expected search time for some $q \neq e^{-1}$, simply divide (12) by ln q.

This measure can be generalized without difficulty to the case where there are two distributions: the actual distribution of the objects in the cells, and the distribution which determines the search strategy (called the "known distribution").

With this generalized measure the information content of distributions generated by a priori probability generators, for example, can be evaluated for efficiency. The actual distribution is, for analytical purposes, given, and the known distribution is the output of the generator.

Predictive mechanisms can be compared by using the generalized measure to compute the marginal increase in the amount of information contributed by the predictive process.

This margin is the difference between the amount of information in the posterior probabilities and the amount of information in the a priori probabilities, given the actual distribution and the observed data. This margin is, in other words, the efficiency of the predictive mechanism.

In the comparision of predictive mechanisms, the independent variable is the type of mechanism. If, instead, the mechanism is held constant and the observed data allowed to vary, the computed margin is the "value" of the observed data. Thus used, the methodology can be applied to measuring in a precise way the contribution of various sensors or competitive tactics in ASW situations.

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